SOLVING ALGEBRAIC INEQUALITIES

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ABSTRACT. We study the problem of solving, possibly quantified, systems of real algebraic equations and inequalities. We propose a way of representing solution sets in a computer algebra system and present an algorithm for computing the representation. We also discuss specialized algorithms for solving several important special cases, including finding "generic solutions", deciding existence of solutions, global optimization of algebraic functions subject to algebraic constraints, and solving linear equation and inequality systems. Finally, we give some examples and present results of some experiments with our implementation of the algorithms within *Mathematica*.

1. INTRODUCTION

Let us first state the main problem in precise terms. To this end, let us explain what do we mean by a system of real algebraic equations and inequalities.

Definition 1.1. *A basic algebraic function given by a polynomial $f(x_1,\ldots,x_n,y)$ and an integer k is the function*

$$Root_{y,k}f:\mathbb{R}^n \ni x_1,\ldots,x_n \longrightarrow Root_{y,k}f(x_1,\ldots,x_n)\in\mathbb{R}$$

where $Root_{y,k}f(x_1,\ldots,x_n)$ is the k-th real root of $f(x_1,\ldots,x_n,y)$ treated as a univariate polynomial in y. The function is defined for those values of $x_1,\ldots,x_n$ for which $f(x_1,\ldots,x_n,y)$ has at least k real roots. The real roots are ordered by the increasing value, counting multiplicities.

A real algebraic function is an arbitrary composition of polynomials, basic algebraic functions, and rational powers. The domain of a real algebraic function f is a set of those points in $\mathbb{R}^n$, for which all basic algebraic functions in f are defined, all negative powers in f have non-zero bases, and all non-integer rational powers in f have non-negative real arguments.

A system of real algebraic equations and inequalities in variables $x_1,\ldots,x_n$ is an alternative of conjunctions of

$$f_k(x_1,\ldots,x_n)\rho_k g_k(x_1,\ldots,x_n)$$

where each ρ_k is one of $<,\leq,\geq,>,=$, or $\neq$, and each f_k and g_k is a real algebraic function. A point $(a_1,\ldots,a_n)\in\mathbb{R}^n$ is a solution of the system if for at least one term of the alternative, the point belongs to the domain of all algebraic functions in this term, and satisfies all the equations and inequalities in this term.

Example 1.2. *We do not require that a solution must belong to domains of all algebraic functions in the entire system . For instance the solution set of*

$$x\geq 0\wedge\sqrt{x}<1\vee x<0\wedge\sqrt{-x}<1$$

is $-1<x<1$, even though only 0 belongs to domains of both radicals.

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Definition 1.3. *A quantified system of real algebraic equations and inequalities in free variables $x_1,\dots,x_n$ and quantified variables $t_1,\dots,t_m$ is a logical formula of the form*

$$Q_1 t_1 \dots Q_m t_m S(t_1,\dots,t_m;x_1,\dots,x_n)$$

Where Q_i is $\exists$ or $\forall$, and S is a system of real algebraic equations and inequalities in $t_1,\dots,t_m,x_1,\dots,x_n$.

By Tarski's theorem (see [14]), solution sets of, possibly quantified, real algebraic equation and inequality systems are semialgebraic. (Algebraic functions can be successively replaced with new variables, and the condition of being the k-th root of a polynomial can be written as a quantified polynomial equation and inequality system.) In particular they need not to be finite so we cannot enumerate them. Instead by solving a system we will mean finding a description of the solution set in some simple and useful standard form not containing quantifiers. We claim that such a simple and useful form is the cylindrical solution form described below, and in the following sections we will give examples showing how the form is useful for instance in global optimization of algebraic functions subject to algebraic constraints, computing multidimensional integrals, and visualization of semialgebraic sets.

Definition 1.4. *A cylindrical form in variables $x_k,\dots,x_n$ with parameters $x_1,\dots,x_{k-1}$ is defined recursively to be*

$$B_1 \wedge C_1 \vee \dots \vee B_m \wedge C_m$$

where C_i is a cylindrical form in variables $x_{k+1},\dots,x_n$ with parameters $x_1,\dots,x_k$, and B_i is one of

$$\begin{array}{ccccc} f(x_1,\dots,x_{k-1}) & \rho & x_k & \sigma & g(x_1,\dots,x_{k-1}) \\ f(x_1,\dots,x_{k-1}) & \rho & x_k & & \\ & & x_k & \sigma & g(x_1,\dots,x_{k-1}) \\ & & x_k & = & g(x_1,\dots,x_{k-1}) \\ & & true & & \end{array}$$

where f and g are basic *algebraic functions, and ρ and σ are $<$ or $\leq$. A cylindrical form in no variables is the Boolean constant true.*

A cylindrical solution form of an equation and inequality system

$$Q_1 t_1 \dots Q_m t_m S(t_1,\dots,t_m;x_1,\dots,x_n)$$

is a cylindrical form in variables $x_1,\dots,x_n$ with no parameters describing the solution set of the system.

In our implementation basic algebraic functions are represented by *Mathematica* Root objects. (See [15], [13]. *Mathematica* 4.0 no longer factors the defining polynomials of non-constant Root objects.) Basic algebraic functions given by polynomials of degree less than three are represented in terms of rational functions and square roots.

Example 1.5. *A cylindrical solution form of $x^2+y^2+z^2<1$ is*

$$-1<x<1 \wedge -\sqrt{1-x^2}<y<\sqrt{1-x^2} \wedge \\ -\sqrt{1-x^2-y^2}<z<\sqrt{1-x^2-y^2}$$

A cylindrical solution form of

$$\exists x : x^8+ax+b=0 \wedge -1<x<1$$

is

$$\begin{gathered} a \le -8 \wedge -1+a < b < -1-a \vee \\ -8 < a \le 0 \wedge -1+a < b \le r(a) \vee \\ 0 < a < 8 \wedge -1-a < b \le r(a) \vee \\ a \ge 8 \wedge -1-a < b < -1+a \end{gathered}$$

where $r(a) = Root_{y,1}(-823543a^8 + 16777216y^7)$.

The Cylindrical Algebraic Decomposition (CAD) algorithm (see [2], [1]) is a constructive proof of the fact that every semialgebraic set, and hence every solution set of a quantified algebraic equation and inequality system, can be represented by a cylindrical solution form. In Section 2 we present an algorithm (based on CAD) allowing to compute such representation. Next, we describe several simpler algorithms that can be used in some important for applications special cases. Finally, we will show some experimental results.

2. THE MAIN ALGORITHM

The input is a quantified system

$$Q_1 t_1 \ldots Q_m t_m S(t_1, \ldots, t_m; x_1, \ldots, x_n)$$

of real algebraic equations and inequalities in free variables $x_1, \ldots, x_n$ and quantified variables $t_1, \ldots, t_m$, where m may be zero. The algorithm computes a cylindrical solution form of the system.

2.1. **Polynomialization.**

First we successively replace algebraic functions with new variables, starting with the innermost basic algebraic functions or radicals. When replacing $Root_{y,k} f(x_1, \ldots, x_n)$ with a new variable z, we add the equation $f(x_1, \ldots, x_n, z) = 0$ to all terms of the alternative S which contained the replaced basic algebraic function. Similarly, we replace a radical $f^{p/q}$ with z^p and add the equation $z^q = f$ and the inequality $f \ge 0$ to all terms of the alternative S which contained the replaced radical. We keep track of which variables replace what algebraic functions and of the order in which they were replaced.

Next, we put all equations and inequalities in the form $f < 0$, $f \le 0$, $f = 0$, or $f \ne 0$, put all rational functions in the "common denominator" form, and replace equations and inequalities involving rational functions using equivalences

$$\begin{aligned} f/g < 0 &\Leftrightarrow f > 0 \wedge g < 0 \vee f < 0 \wedge g > 0 \\ f/g \le 0 &\Leftrightarrow f \ge 0 \wedge g < 0 \vee f \le 0 \wedge g > 0 \\ f/g = 0 &\Leftrightarrow f = 0 \wedge g \ne 0 \\ f/g \ne 0 &\Leftrightarrow f \ne 0 \wedge g \ne 0 \end{aligned}$$

Finally, we put the, now polynomial, equation and inequality system in the disjunctive normal form.

2.2. **Projection.**

This is the projection phase of the CAD algorithm (see [2], [1]). First we project with respect to the variables replacing algebraic function, in the reverse replacement order. Then we project the quantified variables starting with the innermost quantifiers i.e. from t_m to t_1. Finally, we project out free variables x_n through x_2. We can reorder variables within blocks of identical quantifiers and within the free variables if we do not have a preference as to the order of free variables in the solution. In this case we use a heuristics attempting to minimize the size and degrees of the projection. We use two types of projection.

We start with the "short projection". If there are any equational constraints present or if the projected variable replaces an algebraic function, we use the equational constraint case projection suggested in [3]. If there are several equational constraints, we select the pivot (and the projection variable, if we have a choice) based on first whether its all factors have constant leading coefficients, and second on how low is its degree. Equational constraints with nontrivial contents are disqualified. We propagate the remaining equational constraints using the fact that a resultant of equational constraints is an equational constraint. After we run out of replacement variables and equational constraints we continue with the McCallum's projection operator for well-oriented sets of polynomials (see [7], [8]).

The solution form construction phase may fail if the short projection was used, and one of the polynomials of McCallum's projection becomes identically zero on a positive-dimensional cell, or if an equational constraint used as a pivot becomes identically zero on a cell. In this case we use the full projection operator described in [5].

2.3. **Construction of solution form.** Since we are using algebraic functions to describe the solution set we, as opposed to the classical CAD algorithm, do not need to generate and store all the cells before constructing the solution form. Therefore we can use the following recursive algorithm RCSF which, on the k-th recursion level, generates the solution form for the first k variables (in the inverse projection order) belonging to a specified cell. (Remember, in this order the free variables come first, then the quantified variables, outermost quantifiers first, and at the end the variables replacing algebraic functions, in the replacement order.) To generate the full solution we call RCSF on the 0-th recursion level.

Algorithm 2.1. *RCSF*

Input:

- *cell_data* contains information about the cell over which we are constructing the solution. This includes a sample point in the cell, i.e. values for the first k variables, and information whether the cell is zero-dimensional and which (not all) elements of the alternative in S are marked as known to be $false$ on the cell.
- *proj_data* contains all the information from the first two phases of the algorithm, i.e. information about the system of polynomial equations and inequalities S, the variables replacing algebraic functions, the subsequent projection types, projection variables, quantifiers, and sets of projection polynomials.

Output:

- A formula representing solutions of the system, for the first k variables belonging to the input cell. If the $k+1$ -st variable is a free variable, it is a cylindrical form in the free variables left, with the first k variables as parameters. Otherwise it is $true$ or $false$.

(1) Let v be the $k+1$ -st variable.
(2) If v is a variable replacing a basic algebraic function $Root_{y,p} f$, find the real roots of f in y, after replacing the first k variables with coordinates of the sample point. (The chosen projection order guarantees that f after the replacement becomes a univariate polynomial.) If there are at least p roots, counting multiplicities, choose the p-th root as the $k+1$-st coordinate of the sample point, and check if substitution of the new sample point makes any more elements of the alternative in S $false$. Otherwise, choose 0 as the $k+1$-st coordinate and mark all the elements of the alternative in S which contain v as known to be $false$ on the cell. If all the elements of the alternative in S are known to be $false$ on the cell return $false$. If v was the last variable return $true$, else call *RCSF* recursively.

(3) If v is a variable replacing a radical $f^{1/q}$, let a be f with the first k variables replaced with the coordinates of the sample point. The order of projection chosen guarantees that a is a constant. If a is nonnegative, choose the $k+1$-st coordinate of the sample point to be $a^{1/q}$, and check if substitution of the new sample point makes any more elements of the alternative in S $false$. Otherwise choose the coordinate to be 0 (since $f \geq 0$ was included in all elements of the alternative containing v, these elements are already marked as known to be $false$). If all the elements of the alternative in S are known to be $false$ on the cell return $false$. If v was the last variable return $true$, else call *RCSF* recursively.

(4) If v was projected out using an equational constraint let pts be all the real roots of factors of the pivot, after replacing the first k variables with coordinates of the sample point. Otherwise, let pts be a set of all the real roots of the $k+1$-variate projection polynomials, after replacing the first k variables with coordinates of the sample point, and of rational points, one in every interval left after removing the roots from the real line. The algorithm can fail at this point if v was projected out using an equational constraint and any of factors of the pivot becomes identically zero after the substitution, or if the McCallum's projection was used, the cell is not zero-dimensional, and one of the projection polynomials becomes identically zero after the substitution. In this case we start over with the full projection. If the McCallum's projection was used, the cell is zero-dimensional, and one of the projection polynomials becomes identically zero after the substitution we use a suitable derivative instead, as described in [8].

(5) Take the subsequent elements of pts as the $k+1$-st coordinates of the sample point, and determine the solution of the system over the corresponding cell. First check if substitution of the new sample point makes any more elements of the alternative in S $false$. If all the elements of the alternative in S are known to be $false$ on the cell the solution is $false$. If v was the last variable, the solution is $true$. Otherwise we compute the solution calling *RCSF* recursively.

(6) If v is a quantified variable we may return the answer without looking at all the pts. The possible solutions of the system over the cells corresponding to elements of pts are $true$ and $false$, since all the remaining variables are quantified. If v is quantified by the existential quantifier and a solution over the cell corresponding to one of the pts is $true$ we return $true$. If v is quantified by the general quantifier and a solution over the cell corresponding to one of the pts is $false$ we return $false$. If we went through all the pts without returning early we return $false$ for the existential quantifier and $true$ for the general quantifier. Because in this case we may find the answer without looking at all pts we want to try the easier cases first. To this end we take the elements of pts in 5. in order of increasing degrees of their minimal polynomials.

(7) We are left with the case when v is a free variable. The polynomials used in 4. to compute pts are delineable on the cell, therefore the different real roots of these polynomials on the cell are values of non-intersecting basic algebraic functions given by these polynomials. We keep track of the basic algebraic function corresponding to each root, and so we can write the part of answer corresponding to each of the elements of pts as the conjunction of one of $v = f$, $f < v < g$, $v < f$, and $v > f$, and the solution of the system over the cell corresponding to the element, where f and g are the basic algebraic functions corresponding to the appropriate roots. If v was projected out using an equational constraint we return the

alternative of the constructed parts of the answer, else if the solutions of the system over the cells corresponding to adjoining elements of *pts* are identical we join the corresponding parts of the answer and return the alternative of the resulting formulas.

For finding roots of polynomials with algebraic number coefficients we use the algorithm described in [12], modified to compute real roots only. To avoid repeating computations we cache resultant, factorization, and real root isolation computations.

The recursive nature of the algorithm and joining of the cells which can be joined in 7. results in a much lower memory usage of our algorithm than that of the classical CAD algorithm. In practice we have not seen an example in which the space complexity rather than the time complexity would be the main limitation. Step 5. of the algorithm allows parallelization of computations, however our implementation is purely sequential.

3. The full dimensional case

An important category of problems for which the algorithm can be substantially simplified is when the solution set of the inequality system is open, or when we are interested only in the full dimensional part of the solution set. One such case is the decision problem for systems of strong polynomial inequalities, i.e. the problem of deciding whether an open semialgebraic set is nonempty. In this case the original system contains only strong polynomial inequalities and all variables are existentially quantified. In [9] McCallum noticed that it suffices to construct sample points in full dimensional cells only, thus eliminating the need for any algebraic number computations. In [11] the author has shown that in this case we can also use a simpler projection operator. Here we present an extension of this idea to systems containing free variables.

Suppose we have a system of real polynomial inequalities (no equations, algebraic functions, or quantifiers). In some applications, like multidimensional integration or graphical visualization, it may be enough to know a "generic" solution set, which is correct "up to a lower dimensional set". Let us be more precise.

Definition 3.1. *Let S be a system of real polynomial inequalities in n variables, and let $A \subseteq \mathbb{R}^n$ be the solution set of S. A semialgebraic set $B \subseteq \mathbb{R}^n$ is a generic solution set of S if the error set $A \setminus B \cup B \setminus A$ is at most $n-1$-dimensional.*

The following algorithm computes a generic solution set, and gives an at most $n-1$-dimensional set containing the error set.

Algorithm 3.2. *GCAD*

Input:

- A system S of real polynomial inequalities.
- A cylindrical solution formula describing a generic solution set of S, and a set of equations, such that the error set is contained in the union of their zero sets.

Output:

- A cylindrical solution formula describing a generic solution set of S, and a set of equations, such that the error set is contained in the union of their zero sets.

Let us first note that if S contains weak inequalities or inequations, and S' is S with weak inequalities replaced with their strong versions and inequations removed, then a generic solution set B of S' is a generic solution set of S, and the error set of B as a solution of S is contained in the union of the error set of B as a solution of S' and the zero sets of equations corresponding to the weak inequalities and the inequations. Therefore in the following we may assume that S consists only of strong inequalities.

For a set of polynomials *polys*, let $SFRP(polys)$ denote a set of square-free and relatively prime polynomials multiplicatively generating *polys*. For a set of square-free and relatively prime polynomials *qolys*, let $GP(qolys, v)$ denote the projection of *qolys* with respect to v used in [12]. By definition, $GP(qolys, v)$ consists of the leading coefficients, discriminants, and pairwise resultants of *qolys*.

Let S be a system of strong polynomial inequalities in variables $x_1, \ldots, x_n$, transformed to a form with zero right hand sides of inequalities, and let *polys* be the set of left hand sides of inequalities in S. First we compute the set of projections $projs = (pr_n, \ldots, pr_1)$, where $pr_n = SFRP(polys)$ and $pr_k = SFRP(GP(pr_{k+1}, x_{k+1}))$. Then we obtain the solution formula and the equations for the error set calling the following recursive algorithm with $k = 0$.

Algorithm 3.3. *RGCSF*

Input:

- $fprojs = (pr_n, \ldots, pr_{k+1})$
- $eprojs$ is $fprojs$ with $x_1, \ldots, x_k$ replaced with the (rational number) coordinates of a sample point of the k-dimensional cell c over which we are constructing the solution. All polynomials of pr_k have constant non-zero signs on c.
- *ineqs* is S with $x_1, \ldots, x_k$ replaced with the coordinates of the sample point.

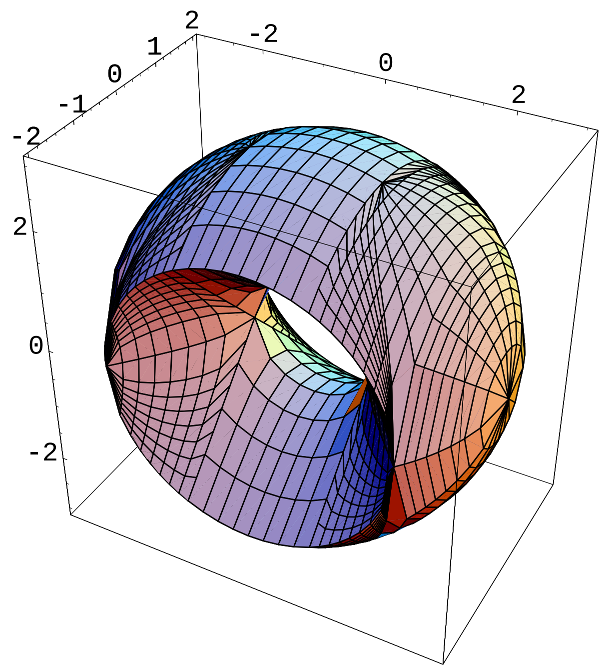
Output:

- A cylindrical form cfm in $x_{k+1}, \ldots, x_n$ with $x_1, \ldots, x_k$ as parameters, and a set *eqns* of polynomials in $x_1, \ldots, x_n$ such that for any point $a = (a_1, \ldots, a_n)$, with $(a_1, \ldots, a_k)$ in the cell, if a is not a zero of any of *eqns* then a is a solution of S iff a is a solution of cfm.

(1) If *ineqs* is $true$ or $false$ we return *ineqs* and no equations.
(2) Let *ps* be the last element of $eprojs$. *ps* is a set of univariate polynomials in x_{k+1}. Isolate roots of *ps*, and find rational numbers pts one in every interval left after removing the roots from the real line.
(3) Polynomials pr_{k+1} are delineable on c. (Their leading coefficients, discriminants, and pairwise resultants have constant non-zero signs, so they have a fixed number of real roots each and the roots do not intersect.) Hence the real roots of these polynomials on c are values of non-intersecting basic algebraic functions given by these polynomials. If we take the subsequent elements of pts as the $k+1$-st coordinates of sample points, we get sample points of $k+1$-dimensional cells on which elements of pr_{k+1} have constant non-zero signs. We call *RGCSF* recursively on each such cell.
(4) We can write the cylindrical form corresponding to each element of pts as a conjunction of one of $f < v < g$, $v < f$, and $v > f$, and the cylindrical form returned by the recursive call, where f and g are the basic algebraic functions corresponding to the appropriate roots. As *eqns* we return all the polynomials given by the recursive calls, and all elements of pr_{k+1} whose root separates two adjoining cells over which S has solutions. If for cells corresponding to adjoining elements of pts the cylindrical forms returned by the recursive calls are identical we join the corresponding cylindrical forms in the answer. We return the alternative of the resulting formulas.

Example 3.4. GCAD *was used by Roger Germundsson in InequalityGraphics package. Here is a graphical representation of the solution set of inequality system*

$$x^2 + y^2 + z^2 \leq 9 \wedge y^2 \leq x^2 + z^2 - 1$$



As another application of the *GCAD* algorithm we can compute the volume of the figure above, which is $64\pi/3$. Here the integration is by far more time consuming than the *GCAD* computation. We can also use the cylindrical solution form to compute the volume numerically. (The results of numerical and symbolic computation do agree.)

4. Global optimization

The problem we investigate in this section is to find the infimum of values of a real algebraic function $f(x_1, \ldots, x_n)$ subject to algebraic equation and inequality constraints $S(x_1, \ldots, x_n)$, and, if possible, find a point in which the infimum is attained. (If the solution set of S is not compact the infimum may not be attained.)

The problem is equivalent to finding the infimum y_{inf} of values of the new variable y on the solution set of the following quantified system of real algebraic equations and inequalities,

$$\exists(x_1, \ldots, x_n) : S(x_1, \ldots, x_n) \wedge y \geq f(x_1, \ldots, x_n)$$

and if possible a point $(a_1, \ldots, a_n)$ satisfying the constraints S and such that $y_{inf} = f(a_1, \ldots, a_n)$.

For this purpose we use a modified version of the Main Algorithm. We use the identical polynomialization and projection phases. The last projection gives us a set of univariate polynomials in y. In step 5. of first call to *RCSF*, with the main variable y, we order *pts* with respect to their increasing values, and call *RCSF* recursively with subsequent elements of *pts* until we get answer *true*. We also modify *RSCF* to return a sample set of values for existentially quantified variables which satisfies the system. If the smallest element of *pts*, for which we get *true* from the recursive call, is a root of one of the projection polynomials, then it is the infimum and the sample point returned by the recursive call is a point at which the infimum is attained. Otherwise the infimum is equal to the largest root of one of the projection polynomials smaller than the element of pts, or $-\infty$ if there are no smaller roots, and the infimum is not attained.

To find an infimum of a polynomial or a rational function subject to strong polynomial inequality constraints (or if we know that the set of points satisfying the constraints is contained in the closure of its interior), we can use a simpler algorithm based on ideas from Section 3. We project out variables $x_1, \ldots, x_n$ using the projection described in Section 3, find rational numbers pts one in every interval left after removing the roots of the last projection from the real line, and then run recursively a modification of *RCSF*, with all $x_1, \ldots, x_n$ existentially quantified, over the subsequent pts in order of increasing values, until we get $true$. Then the infimum is equal to the largest root of one of the projection polynomials smaller than the current element of pts, or $-\infty$ if there are no smaller roots. The modified *RCSF* constructs only full dimensional cells, i.e. in step 4. takes only the rational numbers between the roots of projection polynomials.

5. The linear case

For linear decision problem, i.e. for solving quantified equation and inequality systems with all variables existentially quantified and all equations and inequalities linear we use a method based on the Simplex algorithm from Linear Programming. Since

$$\exists x : a(x) \vee b(x) \Leftrightarrow \exists x : a(x) \vee \exists x : b(x)$$

we may assume that the system is a conjunction of equations and inequalities. We use the following algorithm.

Algorithm 5.1. *LINSIM*

Input:

- A system S which is a conjunction of linear equations and inequalities in $x_1, \ldots, x_k$.

Output:

- $true$ or $false$ depending on whether S has solutions. If the answer is $true$ the algorithm can also give a point $(a_1, \ldots, a_n)$ satisfying the system S.

(1) As in the Simplex algorithm, we add new variables to replace inequalities with equations, with the only difference being that for strong inequalities we require that the new variables be strictly positive.
(2) We Gaussian eliminate the original unrestricted variables. We get a linear system LS_+ of equations in positive and non-negative variables, and an upper-triangular matrix which allows to find values of the original variables, once we have a solution of LS_+.
(3) We use the first phase of the Simplex algorithm to find a solution of LS_+ with all variables nonnegative. If there is no such solution we return $false$.
(4) We look if there is a variable which should be positive, but is zero in the solution. If no we compute the values of the original variables from the matrix in step 2. and return $true$. Otherwise we use the second phase of Simplex algorithm to find the maximal value max of the variable.
(5) If $max = 0$ we return $false$. Otherwise we set the value of the variable to $max/2$ (the solution set of a conjunction of linear equations and inequalities is convex), use the first phase of Simplex algorithm to find non-negative solutions for the remaining variables, and go to step 4.

In [6] Loos and Weispfenning describe an algorithm which allows to eliminate a quantifier if all the equations and inequalities of the system are linear in the quantifier's variable. The result of the elimination however is not given in the cylindrical form. We use this algorithm as a preprocessor to the main algorithm, i.e. we eliminate the innermost quantifiers if the system is linear in their variables and then call the main algorithm on the result. In the last section we compare this approach with the main algorithm without the linear preprocessing. We also use this algorithm whenever we do not insist on the cylindrical form of the result (for example because there are too many free variables in too high degrees).

6. Experimental results

In this section show some experimental results obtained with our implementations of the algorithms presented in this paper. The algorithms were implemented in the C kernel of *Mathematica*. The examples were run on a Pentium II, 233 MHz computer with 64 MB of RAM. The timings are in seconds.

6.1. **The full solution vs. the generic solution.** Here we use strong inequality examples given in [9] . We compare the timings of deciding the problems with all variables existentially quantified (the *DEC* column), finding a cylindrical solution form using the Main Algorithm (the *MAIN* column), and finding a cylindrical solution form for a generic solution set using the *GCAD* algorithm (the *GCAD* column). The cylindrical forms are computed in the (x,y,z) order of variables. The *#c* columns give the total number of cells after putting the solutions given by *MAIN* and *GCAD* in the disjunctive normal form (answer $false$ counts as no cells). The *#e* column gives the number of equations whose solutions cover the error set, as given by the *GCAD* algorithm. Following the notation of [9] let us put.

$$\begin{aligned}
B1 &= x^2+y^2+z^2<1\\
B2 &= (x-1)^2+(y-1)^2+(z-1)^2<1\\
B3 &= (x-1)^2+(y-1)^2+(z+\frac{1}{2})^2<1\\
B4 &= (x-\frac{3}{2})^2+(y-2)^2+z^2<1\\
C1 &= x^2+y^2+z^2+2yz-4y-4z+3<0\wedge\\
&\quad y-1<z\wedge z<y+1\\
C2 &= x^2+y^2+z^2+2yz-4y-4z+3<0\wedge\\
&\quad y+1<z\wedge z<y+2\\
T &= z^4+(2y^2+2x^2+6)z^2+y^4+2x^2y^2-\\
&\quad 10y^2+x^4-10x^2+9<0\\
HB1 &= B1\wedge x+y+z<0\\
HB2 &= B2\wedge x+y+z>3\\
HB3 &= B3\wedge x+y+z<\frac{3}{2}\\
HT &= T\wedge x+y<0
\end{aligned}$$

The decision algorithm uses a heuristic to determine the order of projection, and because of this the timings of *DEC* and *GCAD* may differ even if there are no solutions.

inequalities	*DEC*	*MAIN*	*#c*	*GCAD*	*#c*	*#e*
$B1 \wedge B2$	0.11	3.92	1	0.17	1	0
$B1 \wedge B4$	0.09	0.67	0	0.08	0	0
$B1 \wedge B2 \wedge B3$	0.91	73.05	2	1.32	2	3
$B1 \wedge B2 \wedge B4$	0.92	30.45	0	0.59	0	0
$B1 \wedge C1$	0.11	17.62	1	0.65	1	1
$B1 \wedge C2$	0.16	31.62	0	0.92	0	0
$T \wedge C1$	0.58	774.8	17	5.55	9	8
$T \wedge B2$	0.7	>3000	?	1.1	1	2
$HB1 \wedge HB2 \wedge HB3$	12.23	415	0	3.76	0	0
$HT \wedge HB2 \wedge HB3$	9.72	>3000	?	6.99	0	0
$T \wedge C1 \wedge B2$	2.53	>3000	?	30.79	28	31
$HT \wedge C1 \wedge HB2$	29.2	>3000	?	19.74	0	0

TABLE 1. Strong inequalities

6.2. **Equational constraints.** We compare timings of the Main Algorithm (*MAIN*) and a version of the algorithm which does not use the special case projection using equational constraints (*NOEQC*).

(1) Catastrophe Surface and Sphere from [8]

$$x^2+y^2+z^2=1 \wedge z^3+xz+y=0$$

MAIN returns a solution containing 8 cells after 1.18 seconds. *NOEQC* returns a solution containing 16 cells after 81.26 seconds.

(2) Solotareff's problem from [8]

$$\begin{gathered}\exists b \exists u \exists v : -1<u<v<1 \wedge b<2 \wedge \\ u^4+2u^3-au^2-bu+a+b-3=0 \wedge \\ v^4+2v^3-av^2-bv+a-b+1=0 \wedge \\ 4u^3+6u^2-2au-b=0 \wedge \\ 4v^3+6v^2-2av-b=0\end{gathered}$$

Both algorithms return $a=Root_{y,1}(81y^3-180y^2+448y-432)$, *MAIN* after 1.86 seconds, *NOEQC* after 41.21 seconds.

(3) When the quintic x^5+ax^2+bx+c has at least two different real roots?

$$\exists x \exists y : x^5+ax^2+bx+c=0 \wedge y^5+ay^2+by+c=0 \wedge x \neq y$$

Both algorithms return a solution consisting of 4 cells, *MAIN* after 21 seconds, *NOEQC* after 371 seconds. The solution is

$$\begin{array}{lll} a<0 & \wedge & (b<r(a) \wedge r_1(a,b) \leq c \leq r_2(a,b) \vee \\ & & b=r(a) \wedge r_3(a,b) \leq c \leq r_4(a,b) \vee \\ & & r(a)<b<s(a) \wedge r_1(a,b) \leq c \leq r_2(a,b)) \vee \\ a \geq 0 & \wedge & b<s(a) \wedge r_1(a,b) \leq c \leq r_2(a,b) \end{array}$$

where

$$\begin{aligned} r(a) &= Root_{y,1}(320y^3+27a^4) \\ s(a) &= Root_{y,1}(80y^3-27a^4) \\ r_k(a,b) &= Root_{y,k}(3125y^4+2250a^2by^2-1600ab^3y+ \\ &\quad 108a^5y+256b^5-27a^4b^2) \end{aligned}$$

(4) When the difference between the largest and the smallest root of the cubic x^3+ax+b is at least 1?

$$Root_{y,3}(y^3+ay+b)-Root_{y,1}(y^3+ay+b)\geq 1$$

Both algorithms return

$$a\leq -\tfrac{1}{3}\wedge \tfrac{-2\sqrt{-a^3}}{3\sqrt{3}}\leq b\leq \tfrac{2\sqrt{-a^3}}{3\sqrt{3}}\vee -\tfrac{1}{3}<a<-\tfrac{1}{4}\wedge$$
$$-\tfrac{\sqrt{-4a^3-9a^2-6a-1}}{3\sqrt{3}}\leq b\leq \tfrac{\sqrt{-4a^3-9a^2-6a-1}}{3\sqrt{3}}\vee$$
$$a=-\tfrac{1}{4}\wedge b=0$$

MAIN after 2.46 seconds, *NOEQC* after 9.12 seconds.

6.3. **Global optimization.** Here we use our global optimization algorithm and discuss the use of its two versions, one constructing all cells *ACGO* and the other constructing only the full dimensional cells *FDGO*.

(1) A geometric problem from [10]. Find all values of k for which the inequality

$$a^3+b^3+c^3\geq 3abc+k(a-b)(b-c)(c-a)$$

is true for all a, b, and c sides of a triangle. Since by permutation of a, b, and c we can change the sign of the coefficient at k, we see that the allowable values for k form an interval symmetric with respect to zero. Therefore it is enough to compute the maximum of k for which the inequality is true for all a, b, and c, which is the same as the infimum of k for which there are a, b, and c, such that the opposite inequality is true. So we need to compute the infimum of k on

$$\begin{gathered} a>0\wedge b>0\wedge c>0\wedge k>0\wedge \\ a<b+c\wedge b<c+a\wedge c<a+b\wedge \\ a^3+b^3+c^3<3abc+k(a-b)(b-c)(c-a) \end{gathered}$$

We have strong inequalities only, so we can use *FDGO*, which returns $Root_{y,2}(y^4-72y^2-432)$ after 12.35 seconds. In radicals, the answer is $2\sqrt{9+6\sqrt{3}}$. *ACGO* did not do this example in an hour.

(2) Maximize $\sqrt{x}+\sqrt{y}$ on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$. We can solve the problem directly using *ACGO* in 2.32 seconds, or we can notice that the maximum is equal to the supremum of $x+y$ on $\frac{x^4}{4}+\frac{y^4}{9}<1$, which we can compute using *FDGO* in 0.12 seconds. The answer is $Root_{y,1}(y^{12}-39y^8-465y^4-2197)$, with *ACGO* we also get a point at which the maximum is attained, with *FDGO* we don't.

(3) Find the minimal distance between the largest and the smallest root of the cubic x^3+ax+b assuming the cubic has three real roots and its discriminant is -1. We need to use *ACGO* here. We get the answer $2^{2/3}$ for $a=-2^{-2/3}$ and $b=0$, after 5.58 seconds.

example	*#v*	*#*ln	*cc*	*deg*	*#in*	*dn*s	*MAIN*	*LWPP*
1	3	1	no	2	2	0.3	0.52	0.26
2	3	1	no	2	3	0.3	17.21	9.49
3	3	2	no	2	2	0.5	2.81	0.47
4	3	2	no	2	3	0.5	48.77	7.23
5	2	1	no	3	3	0.5	13.59	3.05
6	2	1	no	3	4	0.5	80.04	12.04
7	3	1	yes	2	3	0.3	0.24	0.13
8	3	1	yes	2	4	0.3	1.09	0.32
9	3	1	yes	2	5	0.3	10.62	24.34
10	3	1	yes	2	7	0.3	35.06	155.62
11	4	2	yes	2	4	0.3	6.12	0.47
12	4	3	yes	2	4	1	16.92	0.45

TABLE 2. Random partially linear systems

6.4. **Linear quantifier elimination as preprocessor.** Here we investigate whether the Loos-Weispfenning linear quantifier elimination algorithm may be useful as a preprocessor to the *MAIN* algorithm. The examples are conjunctions of randomly generated weak polynomial inequalities with all but one variable existentially quantified. The coefficients are rational numbers with 3 decimal digit numerators and denominators. *#v* gives the total number of variables in the system, *#ln* gives the number of linear variables, *deg* is the total degree of the polynomials in the remaining variables, *#in* gives the number of inequalities, and *dns* gives the density of polynomials. In examples 1 through 6 he coefficients at the linear variables are polynomials of total degree *deg* in the remaining variables, in examples 7 through 12 the coefficients at linear variables are constant (this is marked in the *cc* column). The algorithm *LWPP* eliminates the linear variables using the Loos-Weispfenning linear quantifier elimination algorithm and then calls the Main Algorithm. We require that the result be a cylindrical solution form so we have to call the Main Algorithm even if all quantified variables are linear.

The *LWPP* is faster in all examples except of 9 and 10, where the number of inequalities is larger.

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